

# **Structural Deflections-Virtual Work/Unit Load Method**

## **-Dr. E. R. Latifee**

Question: What are **Structural Deflections**?

Answer:

- The deformations or movements of a structure and its components, such as beam ends, truss joints etc., from their original positions/location.
- Deflection is caused by many sources, such as, loads, temperature, construction error, and settlements.
- Deflections are most often caused by internal loadings such as bending moment and axial force.
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**What is the importance of computing deflections in structures?**

Answer:

Structures such as buildings and bridges consist of a number of components such as beams, columns and foundations all of which act together

If beams deflect excessively then this can cause visual distress to the users of the building and can lead to damage of parts of the building including brittle partition dividers between rooms and services such as water and heating pipes and ductwork.

Beam design is carried out according to principles set out in Codes of Practice and typically the maximum deflection is limited to the beam's span length divided by 250. Hence a 5m span beam can deflect as much as 20mm without adverse effect. Thus, in many situations it is necessary to calculate, using numerical methods, the actual beam deflection under the anticipated design load and compare this figure with the allowable value to see if the chosen beam section is adequate.

Computation of deflection of structures is necessary for the following reasons:

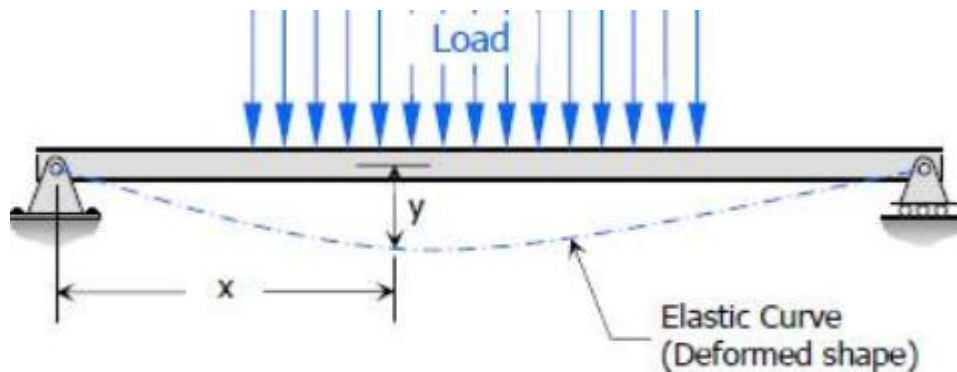
- a. If the deflection of a structure is more than the permissible, the structure will not look aesthetic and will cause psychological upsetting of the occupants.
- b. Excessive deflection may cause cracking in the materials attached to the structure. For example, if the deflection of a floor beam is excessive, the floor finishes and partition walls supported on the beam may get cracked and unserviceable.

- c. The ability to determine the deflection of a structure is very important. It is as important for the designer to determine deflections and strains as these will help to know the stresses caused by the loads.
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An **elastic structure** is one that returns to its original position after the load is removed.

**Define Elastic Curve:** The curved shape of the longitudinal centroidal surface of a beam when the transverse loads acting on it produced wholly elastic stresses.

If the equation for the elastic curve is known, the differential equations of the theory of bending can be used to determine the amount of deflection for any section of a beam, as well as the angle of rotation, the bending moment, and the transverse force. The equation of the elastic curve is derived from the approximate differential equation for the axis of a bent beam, which may be solved by either the analytic or the graphic-analytic method. The latter is particularly convenient when it is sufficient to find the deflection or angle of rotation at isolated points of the beam, in which case there is no need to derive analytic expressions for the elastic curve.



**Figure: Elastic curve**

**Statically indeterminate structures** are the ones where the independent reaction components, and/or internal forces cannot be obtained by using the equations of

equilibrium only. To solve indeterminate systems, we must combine the *concept of equilibrium with compatibility*.

**Check for indeterminacy:**

# of unknowns > # of equations.  $(3m + r) > (3j + c)$ ;  $m$  = # of members,  $r$  = # of reactions,  $j$  = # of joints,  $c$  = # of internal hinges.

**Q. What are the Methods for computation of deflections in structures?**

**Ans:**

1. Double integration method or Macaulay's method
2. Conjugate beam method- The conjugate-beam method was developed by H. Müller-Breslau in 1865.
3. Moment area method
4. Method of elastic weights
5. Virtual work method or unit load method
6. Strain energy method
7. Williot Mohr diagram method
8. Bar-Chain method
9. Castigliano's second theorem method

**Virtual work method or unit load method**

The Deflection,  $\Delta_c = \int \frac{M_0 M_1}{EI} dx$ ,

- Where  $M_0$  = Bending moment distribution (moment at any point) due to actual loading  
 $M_1$  = Bending moment distribution (moment at any point) due to virtual/unit loading  
 $E$  = Modulus of elasticity of the material of beam  
 $I$  = Moment of inertia of beam section  
 $L$  = beam span  
 $\Delta_c$  = deflection at point C

## Procedure for Analysis- Virtual work method or unit load method

The following is a step-by-step procedure to find deflection and slope at a point using the Virtual Work Method. An understanding of Beam Analysis is recommended before undertaking this type of Virtual Work problem.

Step 1: Solve for the support reactions in the real system using Equilibrium and Compatibility.

Step 2: Create an expression(s) for moment in the real system in terms of 'x' distance. The number of expressions will be governed by the number of cuts that are needed to solve the structure.

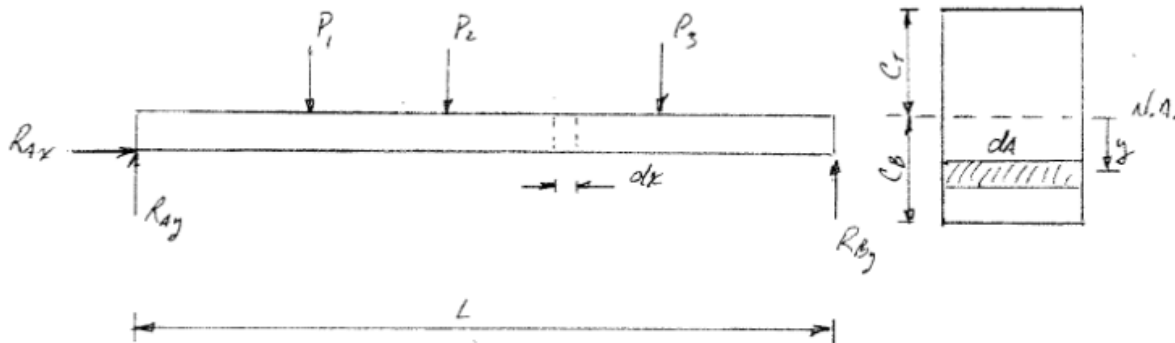
Step 3: Create a virtual system by removing all forces acting on the beam, and applying a unit load to find the expression for deflection, and a unit moment to find an expression for slope. Repeat step 2 for the virtual system.

Step 4: Substitute moment values into the deflection expression and integrate to solve for the deflection at the point of interest.

## Background Theory

### Beam Deflections – Virtual Work

Consider the following



$$M = M(x)$$

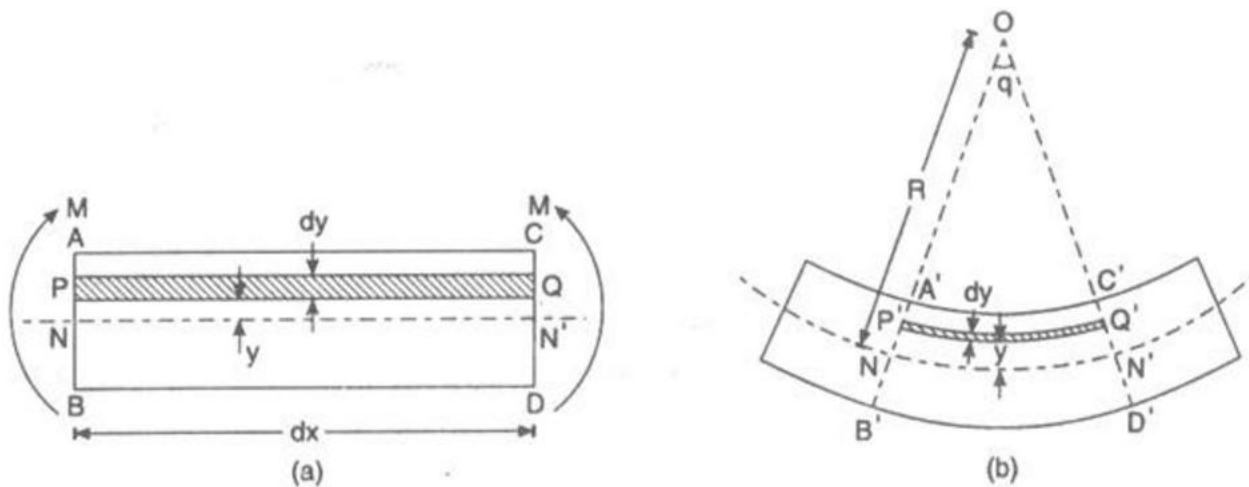
= Moment at any section in the beam due to external loads

$$m = m(x)$$

= Moment at any section in the beam due to a unit action

$$\tilde{\sigma} = \frac{m y}{I}$$

= Stress acting on  $dA$  due to a unit action



The force acting on the differential area  $dA$  due to a unit action is

$$\begin{aligned} \tilde{f} &= \tilde{\sigma} dA \\ &= \left( \frac{m y}{I} \right) dA \end{aligned}$$

The stress due to external loads is

$$\sigma = \frac{M y}{I}$$

The displacement of a differential segment  $dA$  by  $dx$  along the length of the beam is

$$\begin{aligned} \delta &= \epsilon dx \\ &= \left( \frac{\sigma}{E} \right) dx \\ &= \left( \frac{M y}{EI} \right) dx \end{aligned}$$

The work done by the force acting on the differential area  $dA$  due to a unit action as the differential segment of the beam ( $dA$  by  $dx$ ) displaces along the length of the beam by an amount  $\delta$  is

$$\begin{aligned} dW &= \tilde{f} \delta \\ &= \left( \frac{m y}{I} \right) dA \left( \frac{M y}{EI} \right) dx \\ &= \left( \frac{M m y^2}{EI^2} \right) dA dx \end{aligned}$$

The work done within a differential segment (now  $A$  by  $dx$ ) due to a unit action applied to the beam is the integration of the expression above with respect to  $dA$ , i.e.,

$$\begin{aligned} \int_A dW &= \left\{ \int_{c_B}^{c_T} \left( \frac{M m y^2}{EI^2} \right) dA \right\} dx \\ W_{\text{differential segment}} &= \left\{ \left( \frac{Mm}{EI^2} \right) \int_{c_B}^{c_T} y^2 dA \right\} dx \\ &= \left\{ \left( \frac{Mm}{EI^2} \right) I \right\} dx = \left( \frac{Mm}{EI} \right) dx \end{aligned}$$

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The internal work done along the entire length of the beam due to a unit action applied to the beam is the integration of the last expression with respect to  $x$ , i.e.,

$$W_{\text{Internal}} = \int_0^L \left( \frac{M(x) m(x)}{EI} \right) dx$$

The external work done along the entire length of the beam due to a unit action applied to the beam is

$$W_{\text{External}} = (1)(\Delta)$$

With

$$\begin{aligned} W_{\text{External}} &= W_{\text{Internal}} \\ (1)(\Delta) &= \int_0^L \left( \frac{M(x) m(x)}{EI} \right) dx \\ (\Delta) &= \int_0^L \left( \frac{M(x) m(x)}{EI} \right) dx \end{aligned}$$

or the deformation ( $\Delta$ ) of the a beam at the point of application of a unit action (force or moment) is given by the integral on the right.

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