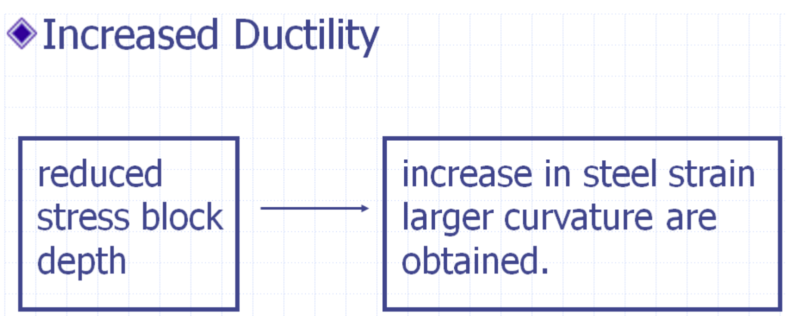
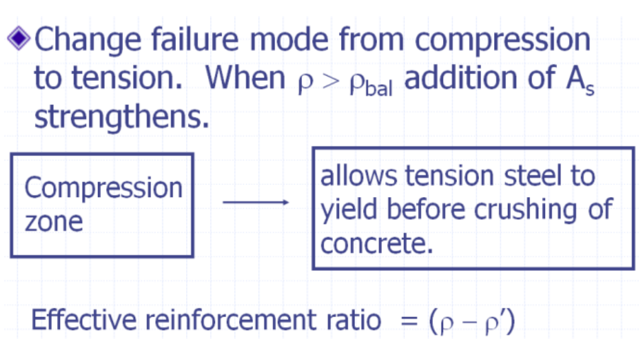
Design of doubly Reinforced Rectangular Beams- Theory with Examples

If a beam cross section is limited because of architectural or other considerations, itmay happen that the concrete cannot develop the compression force required to resist the give bending moment.
In this case, reinforcing is added in the compression zone, resulting in a so-called **doubly reinforced** beam, i.e., one with compression as well as tension reinforcement.
Compression reinforced is also used to improve serviceability, improve long term deflections, and to provide support for stirrups throughout the beam.
Tests of doubly reinforced concrete beams have shown that even if the compression concrete crushes, the beam may very well not collapse if the compression steel is enclosed by stirrups.

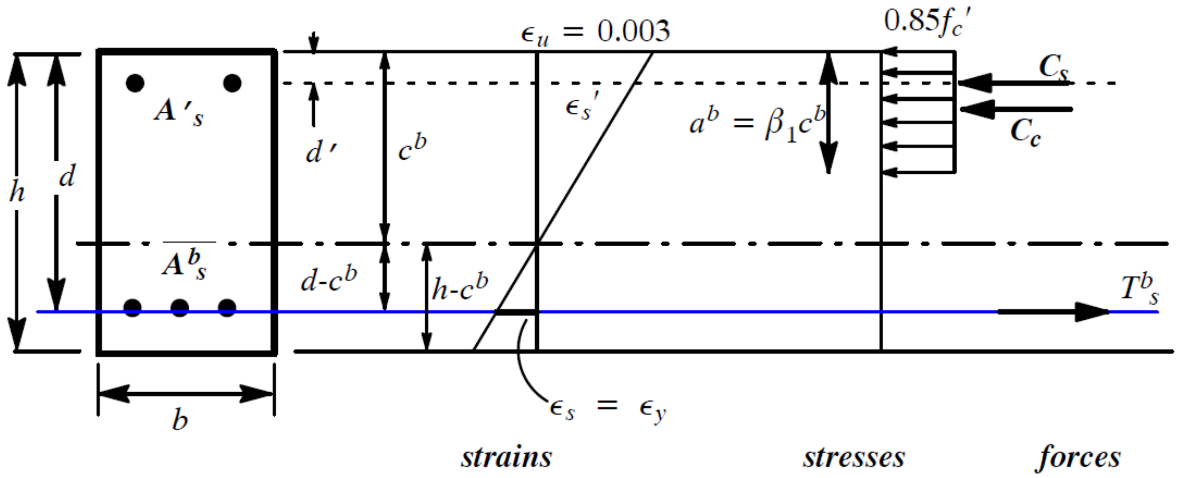
Benefits of doubly reinforced beam

- Eases in Fabrication - Use corner bars to hold & anchor stirrups.
- Reduced sustained load deflections.
  - Creep of concrete in compression zone
  - transfer load to compression steel
  - reduced stress in concrete
  - less creep
  - less sustained load deflection

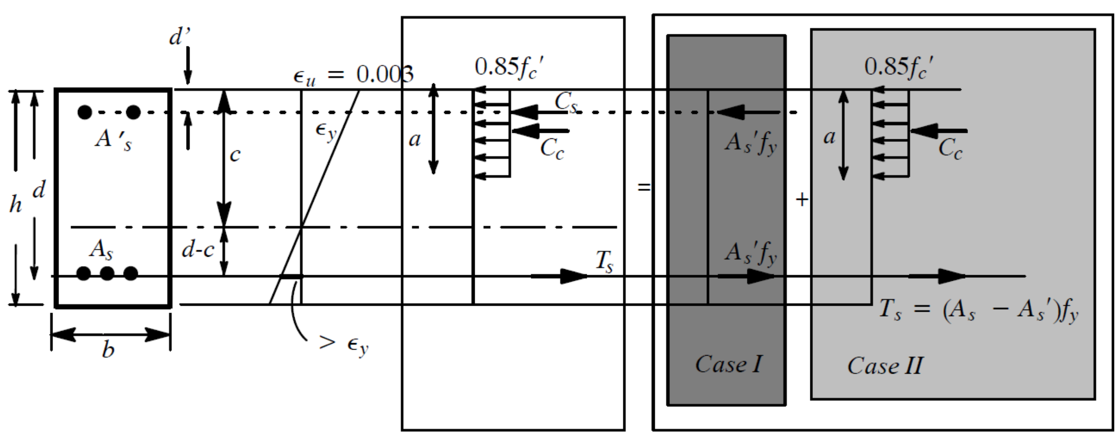


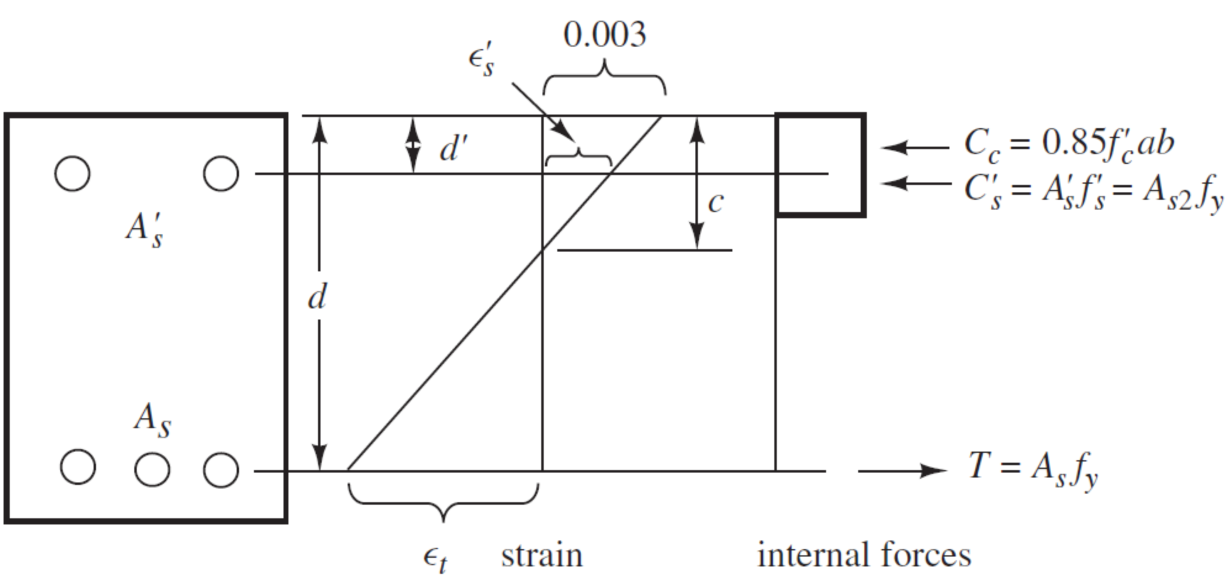


Strength Calculations



Nominal Resisting Moment When Compression Steel Yields





Total resisting moment can be considered as sum of:

1. Moment from corresponding areas of tension and compression steel
2. The moment of some portion of the tension steel acting with concrete.

$$M_n = (A_s - A_s') f_y (d - \frac{\beta_1 c}{2}) + A_s' f_y (d - d')$$

and from equilibrium:

$$0.85 f_c' ab = (A_s - A_s') f_y$$

Solve for "a":

$$a = \frac{A_s - A_s'}{0.85 f_c' b} f_y$$

Compute the design moment strength of the section shown in Figure 5.16 if $f_y = 60{,}000$ psi and $f_c' = 4000$ psi.

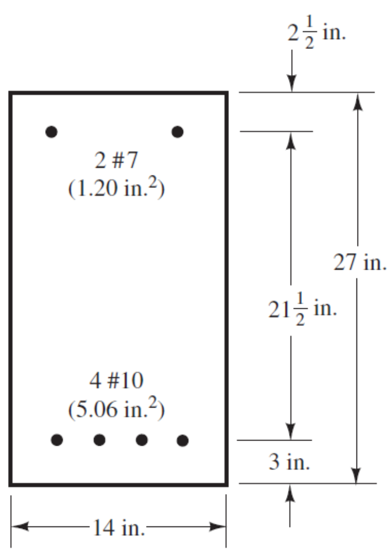


FIGURE 5.16 Beam cross section for Example 5.8.

Writing the Equilibrium Equation Assuming $f'_s = f_y$

$$A_s f_y = 0.85 f'_c b \beta_1 c + A'_s f_y$$

$$(5.06 \text{ in.}^2)(60 \text{ ksi}) = (0.85)(4 \text{ ksi})(14 \text{ in.})(0.85c) + (1.20 \text{ in.}^2)(60 \text{ ksi})$$

$$c = \frac{(5.06 \text{ in.}^2 - 1.20 \text{ in.}^2)(60 \text{ ksi})}{(0.85)(4 \text{ ksi})(0.85)} = 5.72 \text{ in.}$$

$$a = \beta_1 c = (0.85)(5.72 \text{ in.}) = 4.86 \text{ in.}$$

Computing Strains in Compression Steel to Verify Assumption that It Is Yielding

$$\epsilon'_s = \frac{c - d'}{c}(0.003) = \frac{5.72 \text{ in.} - 2.5 \text{ in.}}{5.72 \text{ in.}}(0.003) = 0.00169$$

$$\epsilon_y = \frac{f_y}{E_s} = \frac{60{,}000 \text{ psi}}{29{,}000{,}000 \text{ psi}} = 0.00207 > \epsilon'_s \qquad \therefore f'_s \neq f_y \text{ as assumed}$$

Since the assumption is not valid, we have to use the equilibrium equation that is based on f'_s not yielding.

$$A_s f_y = 0.85 f'_c \beta_1 c b + A'_s \left(\frac{c - d'}{c}\right)(0.003)E_s$$

$$(5.06 \text{ in.}^2)(60 \text{ ksi}) = (0.85)(4 \text{ ksi})(0.85c)(14 \text{ in.}) + (1.20 \text{ in.}^2)\left(\frac{c - 2.5 \text{ in.}}{c}\right)(0.003)(29{,}000 \text{ ksi})$$

Solving the Quadratic Equation for $c = 6.00$ in. and $a = \beta_1 c = 5.10$ in.

Compute strains, stresses, and steel areas

$$\epsilon'_s = \left(\frac{c - d'}{c}\right)(0.003) = \frac{6.00 \text{ in.} - 2.5 \text{ in.}}{6.00 \text{ in.}}(0.003) = 0.00175 < \epsilon_y$$

$$f'_s = \epsilon'_s E_s = (0.00175)(29{,}000 \text{ ksi}) = 50.75 \text{ ksi}$$

$$A_{s2} = \frac{A'_s f'_s}{f_y} = \frac{(1.20 \text{ in.}^2)(50{,}750 \text{ psi})}{60{,}000 \text{ psi}} = 1.015 \text{ in.}^2$$

$$A_{s1} = A_s - A_{s2} = 5.06 \text{ in.}^2 - 1.015 \text{ in.}^2 = 4.045 \text{ in.}^2$$

$$\epsilon_t = \left(\frac{d - c}{c}\right)(0.003) = \frac{24 \text{ in.} - 6.00 \text{ in.}}{6.00 \text{ in.}}(0.003) = 0.0090 > 0.005 \qquad \therefore \phi = 0.9$$

Then the design moment strength is

$$\phi M_n = \phi\left[A_{s1} f_y \left(d - \frac{a}{2}\right) + A'_s f'_s (d - d')\right]$$

$$= 0.9\left[(4.045 \text{ in.}^2)(60 \text{ ksi})\left(24 \text{ in.} - \frac{5.10 \text{ in.}}{2}\right) + (1.20 \text{ in.}^2)(50.75 \text{ ksi})(24 \text{ in.} - 2.5 \text{ in.})\right]$$

$$= 5863 \text{ in-k} = 488.6 \text{ ft-k}$$

Design of Doubly reinforced Beam

Sufficient tensile steel can be placed in most beams so that compression steel is not needed.

Example 1

A beam is limited to the dimensions $b = 15$ in., $d = 20$ in., and $d' = 4$ in. If $M_D = 170$ ft-k, $M_L = 225$ ft-k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi, select the reinforcing required.

SOLUTION

$$M_u = (1.2)(170 \text{ ft-k}) + (1.6)(225 \text{ ft-k}) = 564 \text{ ft-k}$$

Assuming $\phi = 0.90$

$$M_n = \frac{564 \text{ ft-k}}{0.90} = 626.7 \text{ ft-k}$$

$$\text{Max } A_{s1} = (0.0181)(15 \text{ in.})(20 \text{ in.}) = 5.43 \text{ in.}^2$$

$$\text{For } \rho = 0.0181 \frac{M_u}{\phi b d^2} = 912.0 \text{ psi (from Appendix A, Table A.13)}$$

$$M_{u1} = (912 \text{ psi})(0.90)(15 \text{ in.})(20 \text{ in.})^2 = 4{,}924{,}800 \text{ in-lb} = 410.4 \text{ ft-k}$$

$$M_{n1} = \frac{410.4 \text{ ft-k}}{0.90} = 456.0 \text{ ft-k}$$

$$M_{n2} = 626.7 \text{ ft-k} - 456.0 \text{ ft-k} = 170.7 \text{ ft-k}$$

Checking to See If Compression Steel Has Yielded

$$a = \frac{A_{s1} f_y}{0.85 f'_c b} = \frac{(5.43 \text{ in.}^2)(60 \text{ ksi})}{(0.85)(4 \text{ ksi})(15 \text{ in.})} = 6.39 \text{ in.}$$

$$c = \frac{6.39 \text{ in.}}{0.85} = 7.52 \text{ in.}$$

$$\epsilon'_s = \left(\frac{7.52 \text{ in.} - 4.00 \text{ in.}}{7.52 \text{ in.}}\right)(0.003) = 0.00140 < \frac{60 \text{ ksi}}{29{,}000 \text{ ksi}} = 0.00207$$

$$\therefore f'_s = (0.00140)(29{,}000 \text{ ksi}) = 40.6 \text{ ksi}$$

$$\text{Theoretical } A'_{s\ \text{reqd}} = \frac{M_{n2}}{f'_s(d - d')}$$

$$= \frac{(12\ \text{in/ft})(170.7\ \text{ft-k})}{(40.6\ \text{ksi})(20\ \text{in.} - 4\ \text{in.})} = \underline{\underline{3.15\ \text{in.}^2}} \qquad \underline{\underline{\text{Try 4 \#8 } (3.14\ \text{in.}^2)}}$$

$$A'_s f'_s = A_{s2} f_y$$

$$A_{s2} = \frac{(3.14\ \text{in.}^2)(40.6\ \text{ksi})}{60\ \text{ksi}} = 2.12\ \text{in.}^2$$

$$A_s = A_{s1} + A_{s2} = 5.43\ \text{in.}^2 + 2.12\ \text{in.}^2 = \underline{\underline{7.55\ \text{in.}^2}} \qquad \underline{\underline{\text{Try 6 \#10 } (7.59\ \text{in.}^2)}}$$

Subsequent checks using actual steel areas reveal $\epsilon_t = 0.00495$, $\phi = 0.896$, and $\phi M_n = 459.4$ ft-k, which is less than M_u by about 0.1%.

Example 2

Design a rectangular beam for $M_D = 325$ ft-k and $M_L = 400$ ft-k if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. The maximum permissible beam dimensions are shown in Figure 5.17.

$$M_u = (1.2)(325\ \text{ft-k}) + (1.6)(400\ \text{ft-k}) = 1030\ \text{ft-k}$$

SOLUTION

Assuming $\phi = 0.90$

$$M_n = \frac{M_u}{\phi} = \frac{1030\ \text{ft-k}}{0.90} = 1144.4\ \text{ft-k}$$

Assuming maximum possible tensile steel with no compression steel and computing beam's nominal moment strength

$$\rho_{max}\ (\text{from Appendix A, Table A.7}) = 0.0181$$

$$A_{s1} = (0.0181)(15\ \text{in.})(28\ \text{in.}) = 7.60\ \text{in.}^2$$

$$\text{For } \rho = 0.0181\ \frac{M_u}{\phi b d^2}\ (\text{from Table A.13}) = 912.0\ \text{psi}$$

$$M_{u1} = (912.0\ \text{psi})(0.9)(15\ \text{in.})(28\ \text{in.})^2 = 9{,}652{,}608\ \text{in-lb}$$

$$= 804.4\ \text{ft-k}$$

$$M_{n1} = \frac{804.4}{0.90} = 893.8\ \text{ft-k}$$

$$M_{n2} = M_n - M_{n1} = 1144.4\ \text{ft-k} - 893.8\ \text{ft-k} = 250.6\ \text{ft-k}$$

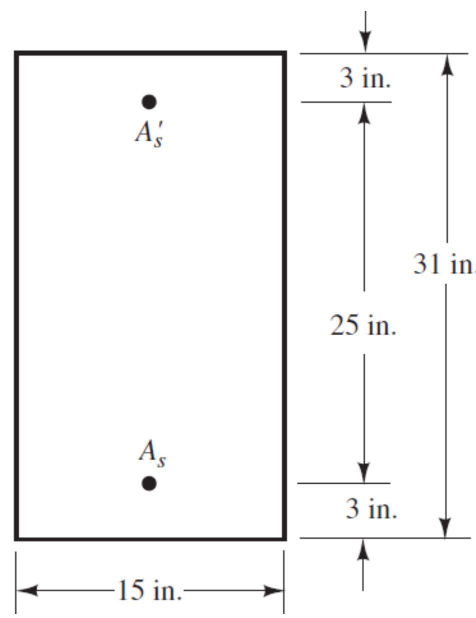
Checking to See Whether Compression Steel Has Yielded

$$a = \frac{(7.60 \text{ in.}^2)(60 \text{ ksi})}{(0.85)(4 \text{ ksi})(15 \text{ in.})} = 8.94 \text{ in.}$$

$$c = \frac{8.94 \text{ in.}}{0.85} = 10.52 \text{ in.}$$

$$\epsilon_s' = \left(\frac{10.52 \text{ in.} - 3 \text{ in.}}{10.52 \text{ in.}}\right)(0.00300) = 0.00214 > 0.00207$$

Therefore, compression steel has yielded.



$$\text{Theoretical } A_s' \text{ required} = \frac{M_{n2}}{(f_y)(d - d')} = \frac{(12 \text{ in/ft})(250.6 \text{ ft-k})}{(60 \text{ ksi})(28 \text{ in.} - 3 \text{ in.})} = 2.00 \text{ in.}^2$$

$$A_s' f_s' = A_{s2} f_y$$

$$A_{s2} = \frac{A_s' f_s'}{f_y} = \frac{(2.00 \text{ in.}^2)(60 \text{ ksi})}{60 \text{ ksi}} = 2.00 \text{ in.}^2 \qquad \underline{\underline{\text{Try 2 \#9 } (2.00 \text{ in.}^2)}}$$

$$A_s = A_{s1} + A_{s2}$$

$$A_s = 7.60 \text{ in.}^2 + 2.00 \text{ in.}^2 = \underline{\underline{9.60 \text{ in.}^2}} \qquad \underline{\underline{\text{Try 8 \#10 } (10.12 \text{ in.}^2)}}$$

If we had been able to select bars with exactly the same areas as calculated here, ϵ_t would have remained $= 0.005$ as originally assumed and $\phi = 0.90$, but such was not the case.

From the equation for c in Section 5.7, c is found to equal 11.24 in. and $a = \beta_1 c = 9.55$ in. using actual, not theoretical, bar areas for A_s and A_s'.

$$\epsilon_s' = \left(\frac{11.24 \text{ in.} - 3 \text{ in.}}{11.24 \text{ in.}}\right)(0.003) = 0.00220 > 0.00207 \text{ compression steel yields}$$

$$\epsilon_t = \left(\frac{28 \text{ in.} - 11.24 \text{ in.}}{11.24 \text{ in.}}\right)(0.003) = 0.00447 < 0.005$$

$$\phi = 0.65 + (0.00447 - 0.002)\left(\frac{250}{3}\right) = 0.855$$

$$\phi M_n = 0.855\left[(10.12 \text{ in.}^2 - 2.00 \text{ in.}^2)(60 \text{ ksi})\left(28 \text{ in.} - \frac{9.55 \text{ in.}}{2}\right) + (2.00 \text{ in.}^2)(60 \text{ ksi})(25 \text{ in.})\right]$$

$$= 12{,}241 \text{ in-lb} = 1020 \text{ ft-k} < 1030 \text{ ft-k} \qquad \underline{\underline{\text{No good}}}$$

The beam does not have sufficient capacity because of the variable ϕ factor. This can be avoided if you are careful in picking bars. Note that the actual value of A_s' is exactly the same as the theoretical value. The actual value of A_s, however, is higher than the theoretical value by $10.12 - 9.6 = 0.52$ in.^{2}. If a new bar selection for A_s' is made whereby the actual value of A_s' exceeds the theoretical value by about this much (0.52 in.^{2}), the design will be adequate. Select three #8 bars ($A_s' = 2.36$ in.^{2}) and repeat the previous steps. Note that the actual steel areas are used below, not the theoretical ones. As a result, the values of c, a, ϵ_s', and f_s' must be recalculated.

Assuming $f_s' = f_y$

$$c = \frac{(A_s - A_s')f_y}{0.85 f_c' b \beta_1} = \frac{(10.12\text{ in.}^2 - 2.36\text{ in.}^2)(60\text{ ksi})}{(0.85)(4\text{ ksi})(15\text{ in.})(0.85)} = 10.74\text{ in.}$$

$$\epsilon_s' = \left(\frac{c - d'}{c}\right)(0.003) = \frac{10.74\text{ in.} - 3\text{ in.}}{10.74\text{ in.}}(0.003) = 0.00216 > \epsilon_y \qquad \therefore \text{ Assumption is valid}$$

$$\epsilon_t = \left(\frac{d - c}{c}\right)(0.003) = \frac{28\text{ in.} - 10.74\text{ in.}}{10.74\text{ in.}}(0.003) = 0.00482 < 0.005 \qquad \therefore \phi \neq 0.9$$

$$\phi = 0.65 + (\epsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.88$$

$$A_{s2} = \frac{A_s' f_s'}{f_y} = \frac{(2.36\text{ in.}^2)(60\text{ ksi})}{60\text{ ksi}} = 2.36\text{ in.}^2$$

$$A_{s1} = A_s - A_{s2} = 10.12\text{ in.}^2 - 2.36\text{ in.}^2 = 7.76\text{ in.}^2$$

$$M_{n1} = A_{s1} f_y \left(d - \frac{a}{2}\right) = (7.76\text{ in.}^2)(60\text{ ksi})\left[28\text{ in.} - \frac{(0.85)\,(10.74\text{ in.})}{2}\right] = 10{,}912\text{ in-k} = 909.3\text{ ft-k}$$

$$M_{n2} = A_{s2} f_y (d - d') = (2.36\text{ in.}^2)(60\text{ ksi})(28\text{ in.} - 3\text{ in.}) = 3540\text{ in-k} = 295\text{ ft-k}$$

$$M_n = M_{n1} + M_{n2} = 909.3\text{ ft-k} + 295\text{ ft-k} = 1204.3\text{ ft-k}$$

$$\phi M_n = (0.88)(1204.3\text{ ft-k}) = 1059.9\text{ ft-k} > M_u \qquad \underline{\underline{\text{OK}}}$$

Note that eight #10 bars will not fit in a single layer in this beam. If they were placed in two layers, the centroid would have to be more than 3 in. from the bottom of the section. It would be necessary to increase the beam depth, h, in order to provide for two layers or to use bundled

Summary of Design Procedure

1. Determine the design moment (ϕM_{n1}) that the beam can carry using maximum amount of tension reinforcement, as a tension-controlled section.
2. If ϕM_{n1} is larger in magnitude than the total factored moment M_u, then the section is to be designed as singly reinforced (dealt with before).
3. If ϕM_{n1} is smaller than M_u, then the section is to be designed as doubly reinforced. The remaining moment, $M_u - \phi M_{n1}$ is to be resisted by a couple resulting from forces in compression steel and the additional tension steel.
4. The magnitude of the additional tensile force in the reinforcement is evaluated as

$$T_2 = \frac{M_u - \phi M_{n1}}{d - d'}$$

5. Determine the area of additional tension reinforcement, $A_{s2} = \dfrac{T_2}{f_y}$.

5. Determine the total area of tension reinforcement, $A_s = A_{s,\,max} + A_{s2}$

6. Determine the area of compression reinforcement as

$$A'_s = T_2\left(\frac{f_y}{f'_s}\right)$$

7. Check the moment strength of the cross section.